Remediating Non-Positive Definite State Covariances for Collision Probability Estimation

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Agenda

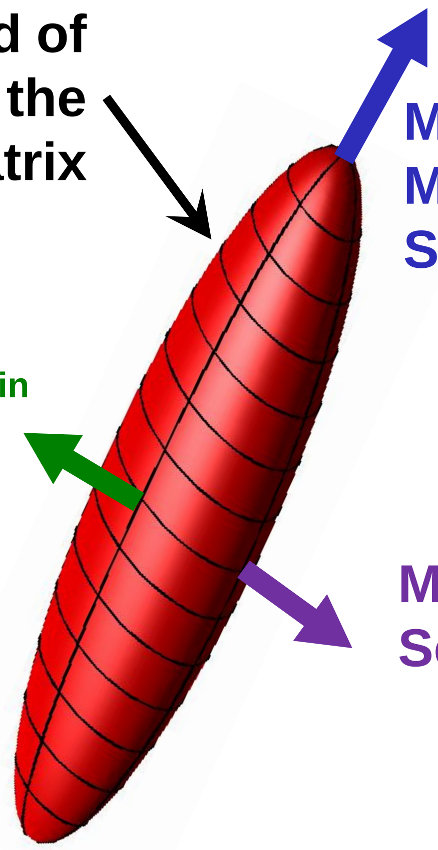
- **Introduction**
  - Motivation and objectives
  - Geometrical visualizations of covariances
- **The observed frequency of NPD covariances**
  - Analysis of 830,000 actual conjunctions spanning 2 years
  - Low-precision vs. high-precision covariances
  - Low-eccentricity vs. high-eccentricity orbits
- **Covariance remediation for Pc estimation**
  - Covariance requirements for Pc-related calculations
  - *Spectrum shifting*, *Higham*, and *eigenvalue clipping* methods
  - Pc estimation with *eigenvalue clipping* remediation

Motivation and Objectives

- **Motivation:**
  - **The probability of collision (Pc) between two Earth-orbiting satellites requires estimates of their orbital trajectories and associated uncertainties**
  - **Pc estimation requires processing conjunction state vectors and covariance matrices**
  - **CARA sometimes encounters non-positive definite (NPD) covariances that can potentially prevent or adversely affect Pc estimation**
- **Objectives:**
  - **Investigate the frequency of NPD covariances**
  - **Implement method(s) to remediate NPD covariances when necessary**

Geometrical Visualization of a 3x3 Covariance Matrix

The 1-sigma ellipsoid of the PDF defined by the 3x3 covariance matrix



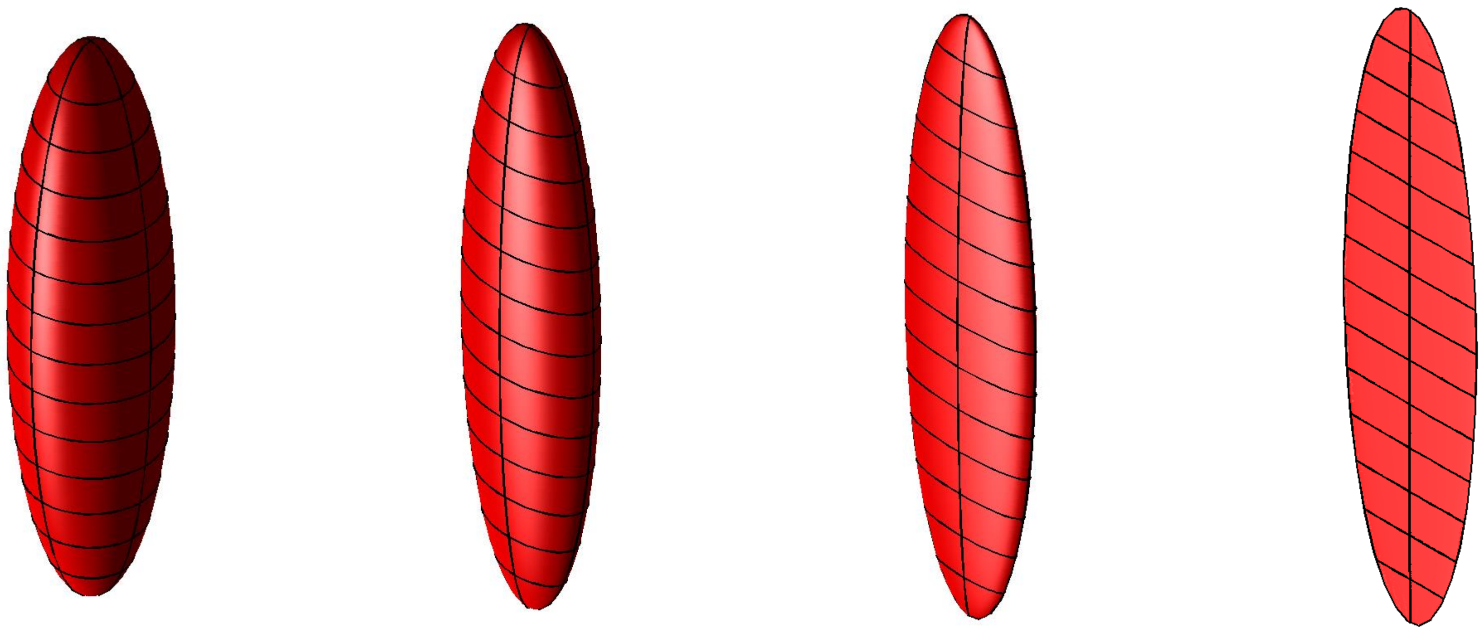
Major Eigenvector = V_{max}
Major Eigenvalue = λ_{max}
Semi-major axis = $\sigma_{max} = (\lambda_{max})^{1/2}$

Minor Eigenvector = V_{min}
Semi-minor axis = σ_{min}

Medium Eigenvector = V_{med}
Semi-principal axis = σ_{med}

Transition from Positive Definite to Semi-Positive Definite 3x3 Covariances

$\lambda_{min} = \lambda_{med}$ $\sigma_{min} = \sigma_{med}$	$\lambda_{min} = \lambda_{med} / 4$ $\sigma_{min} = \sigma_{med} / 2$	$\lambda_{min} = \lambda_{med} / 100$ $\sigma_{min} = \sigma_{med} / 10$	$\lambda_{min} = 0$ $\sigma_{min} = 0$



Positive definite (PD) covariances have positive σ_{min} values
Positive semi-definite (PSD) covariances have zero σ_{min} values
Non-positive definite (NPD) covariances have imaginary σ_{min} values

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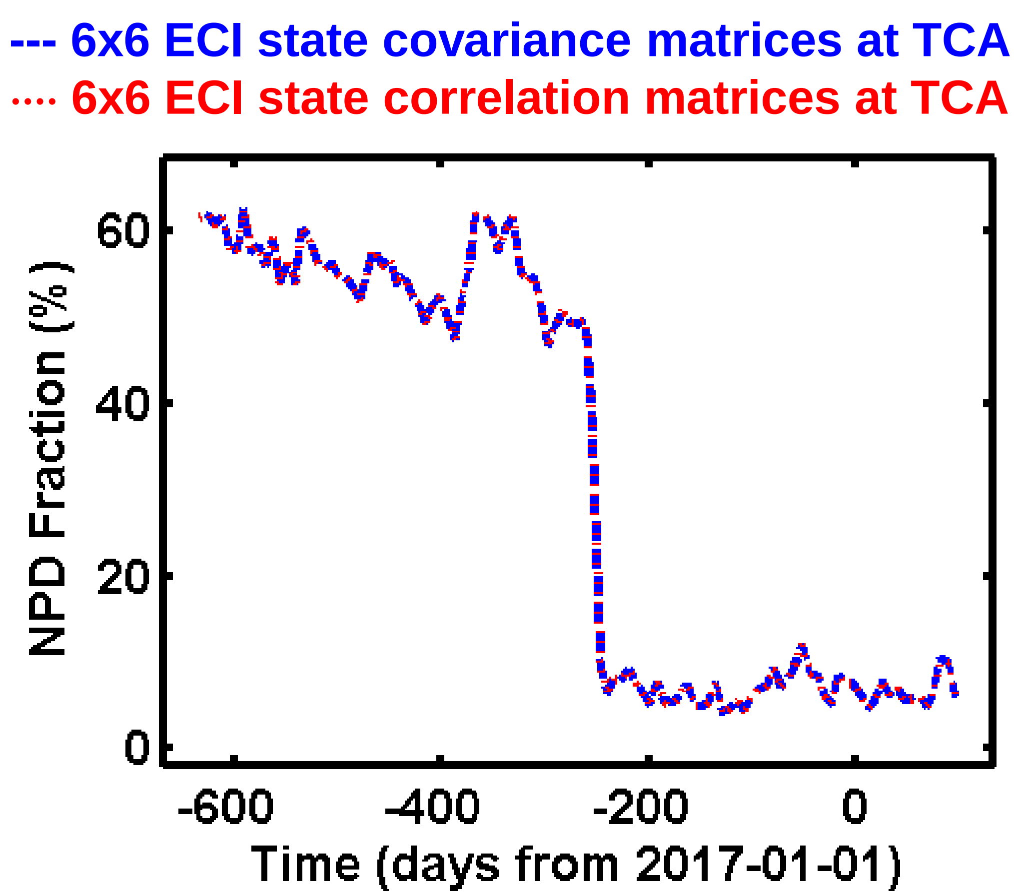
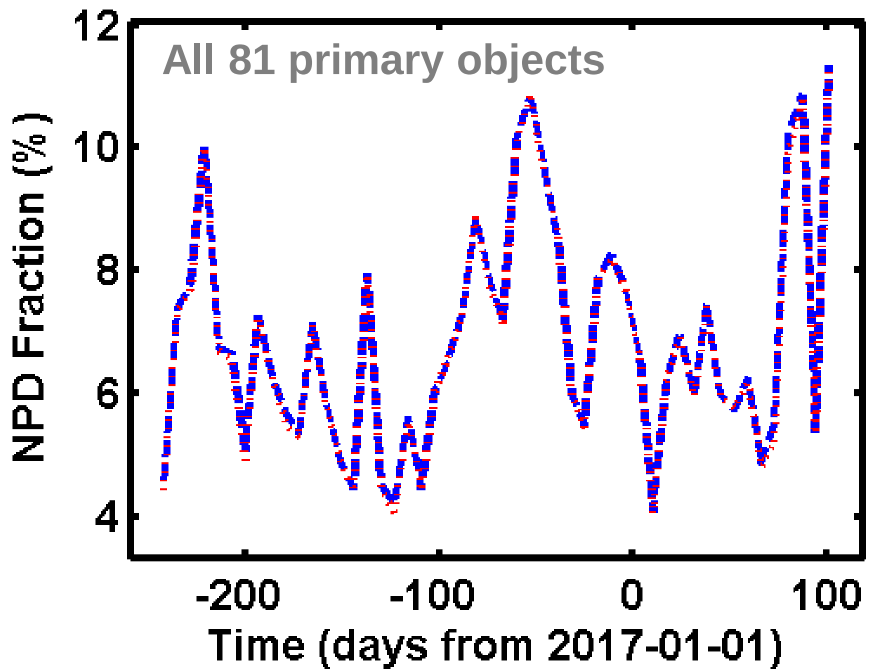
The Frequency of NPD State Covariance Matrices Over Time

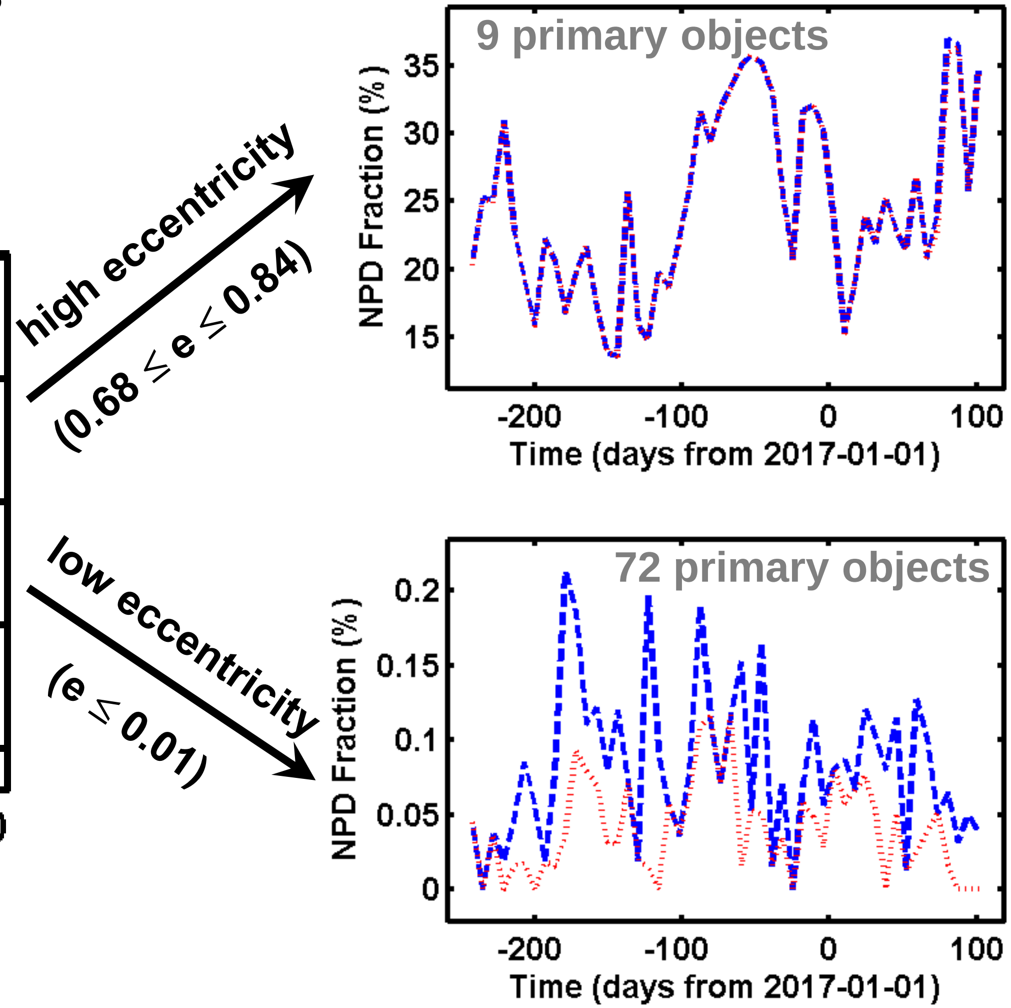
- **Analysis of 830,509 events**
  - **2015-04-01 to 2017-04-06**
- **NPD covariances for primary objects decreased markedly just before 2016-05-01**
  - **Coincides with an increase in the number of significant figures used for covariances**
  - **Better precision leads to fewer NPD state covariances**
- **This same pattern is seen for 3x3 position covariances**
  - **But at frequencies reduced by a factor of 100 or more**



The Frequency of NPD State Covariances: High Eccentricity vs. Low Eccentricity

- **Analysis of 428,589 conjunctions**
  - **2016-05-01 to 2017-04-06**
  - **High precision covariances**





--- 6x6 ECI state covariance matrices at TCA

···· 6x6 ECI state correlation matrices at TCA

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Covariance Requirements for Numerical Stability in Pc-Related Estimations

- **Mahalanobis distance estimation**
  - The marginalized 3x3 relative position covariance $A(t) = A_p(t) + A_s(t)$ needs to be positive definite
- **2D Pc estimation**
  - A marginalized 2x2 projection of $A(t_{ca}) = A_p(t_{ca}) + A_s(t_{ca})$ needs to be positive definite
- **3D Pc estimation**
  - The marginalized 3x3 relative position covariances $A(t_i)$ need to be positive definite at all ephemeris times t_i used in the calculation
- **Monte Carlo Pc estimation**
  - The full NxN state covariances at the sampling epoch time $C(t_{ep})$ need to be at least positive semi-definite

Pc-related calculations don't always require fully PD state covariances for both objects, because they often use combined and marginalized covariances with reduced dimensions

Three Methods Studied for Remediating Non-Positive Definite Covariance Matrices

	Spectrum Shifting	Higham Remediation*	Eigenvalue Clipping
Description	Add an offset to all eigenvalues, and then reconstruct covariances using original eigenvectors	Find closest PSD covariance and/or correlation matrix in terms of Frobenius norm	Clip the eigenvalues at a minimum limit, and then reconstruct covariances using original eigenvectors
Advantages	-Relatively simple to implement	-Mathematically well defined, as used by the financial industry -Algorithms and codes posted on-line	-Simplest to implement -Constrained by physics-based considerations - Produces fully PD position covariances
Dis-advantages	-Assumes original eigenvectors can be used for matrix reconstruction -Offset applied to all eigenvalues, even if they don't need remediation -Not constrained using any physics-based considerations	-Most complicated to implement in software -Only designed to produce PSD matrices -The "closest Frobenius norm" criterion, while well mathematically defined, is not a physics-based criterion	-Assumes original eigenvectors can be used for matrix reconstruction

*N.J. Higham, "Computing the Nearest Correlation Matrix—A Problem from Finance," *IMA Journal of Numerical Analysis* 22, 329-343, 2002.

Pc Estimates for Conjunctions with Remediated NPD Covariances

25338_conj_32035_20161116_093255_20161112_231203

TCA 6x6 ECI covariance status: C_p = PD C_s = NPD C_p+C_s = PD

Remediation Method	2D Pc
No covariance remediation	1.1137×10^{-4}
Spectrum shifting remediation	1.1137×10^{-4}
Higham remediation	1.1137×10^{-4}
Eigenvalue clipping	1.1137×10^{-4}

Most often, remediating 6x6 ECI covariances changes Pc negligibly

***Primary: NOAA 15 Secondary: BREEZE-M DEB**

Pc Estimates for Conjunctions with Remediated NPD Covariances

29499_conj_38481_20160727_045045_20160815_125105*

TCA 6x6 ECI covariance status: C_p = NPD C_s = NPD C_p+C_s = NPD

Remediation Method	2D Pc
No covariance remediation	1.3217×10^{-3}
Spectrum shifting remediation	1.3217×10^{-3}
Higham remediation	1.3464×10^{-3}
Eigenvalue clipping	1.3217×10^{-3}

This ~1.8% Pc difference was the largest seen among 430,000 events

***Primary: METOP-A   Secondary: COSMOS 2251 DEB**

Remediating NPD Position Covariances using the Eigenvalue Clipping Method

$$\text{Eigen-decomposition:} \quad \mathbf{C} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{V}_1 & \rightarrow \\ \leftarrow & \mathbf{V}_2 & \rightarrow \\ \leftarrow & \mathbf{V}_3 & \rightarrow \end{bmatrix}$$

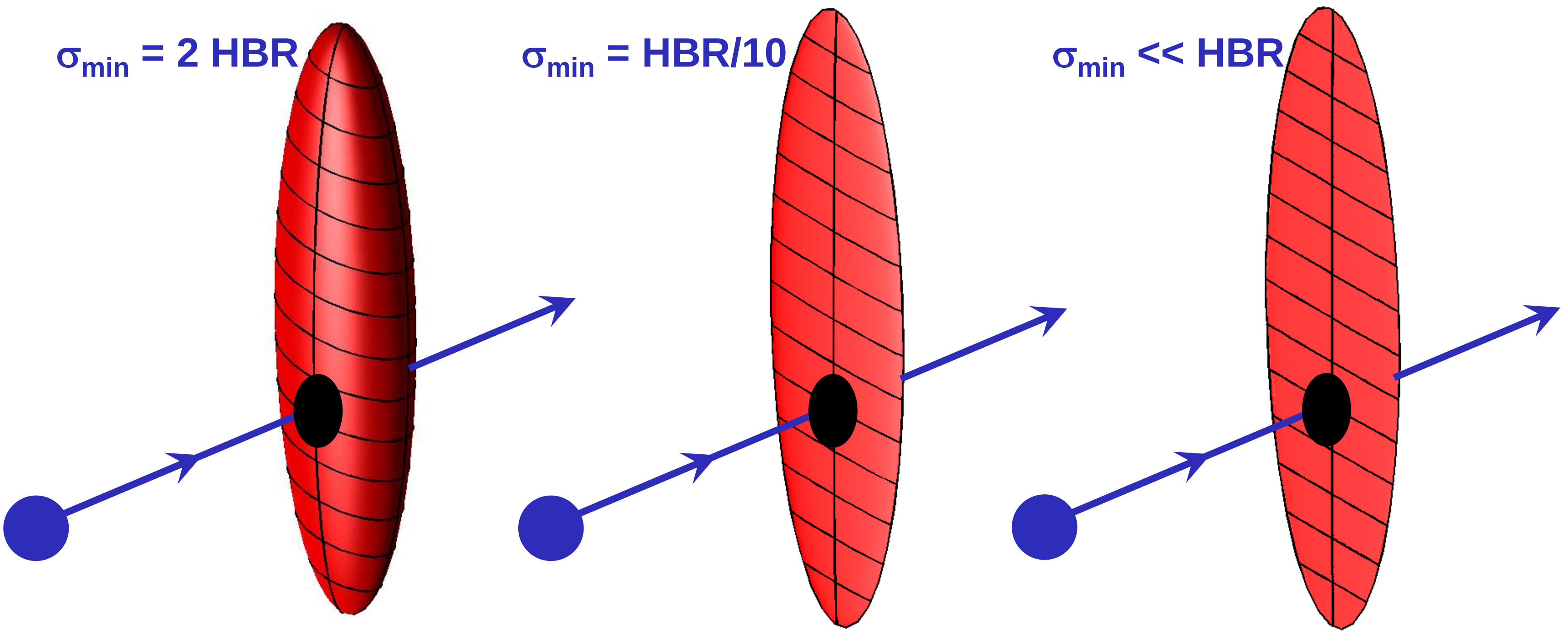
$$\text{Clipped eigenvalues:} \quad \lambda_{i,rem} = \max[\lambda_i, \lambda_{clip}] \qquad \lambda_{clip} \geq 0 = \begin{cases} = 0 & \text{Remediation to PSD} \\ > 0 & \text{Remediation to PD} \end{cases}$$

$$\text{Remediated matrix:} \quad \mathbf{C}_{rem} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_{1,rem} & 0 & 0 \\ 0 & \lambda_{2,rem} & 0 \\ 0 & 0 & \lambda_{3,rem} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{V}_1 & \rightarrow \\ \leftarrow & \mathbf{V}_2 & \rightarrow \\ \leftarrow & \mathbf{V}_3 & \rightarrow \end{bmatrix}$$

What is a sensible value for λ_{clip} - the eigenvalue clipping limit?
Can physics be used to constrain this value?

Collision Probability as a Function of σ_{min}

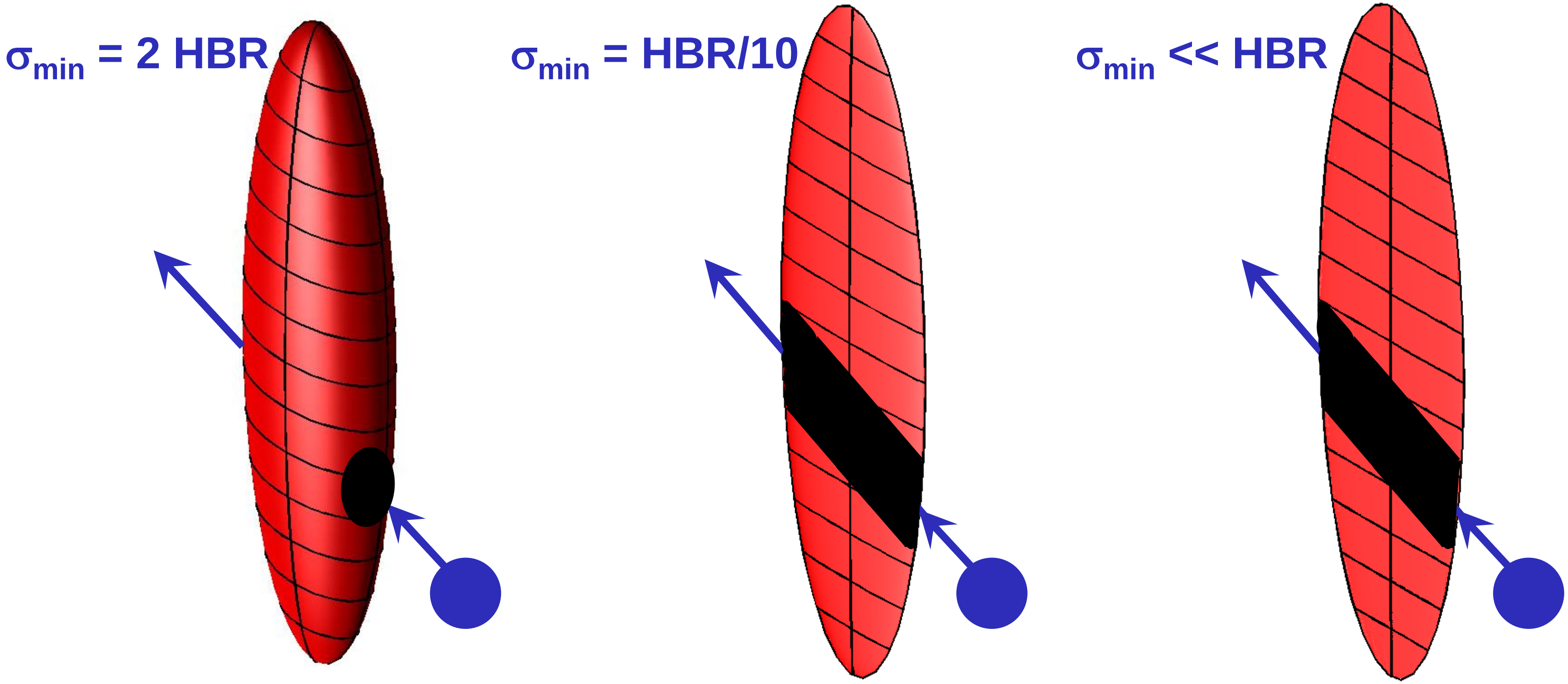
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



These three conjunctions produce similar Pc values (because they've carved out similar fractions of the PDF)

Collision Probability Visualization: Pc as a Function of σ_{min}

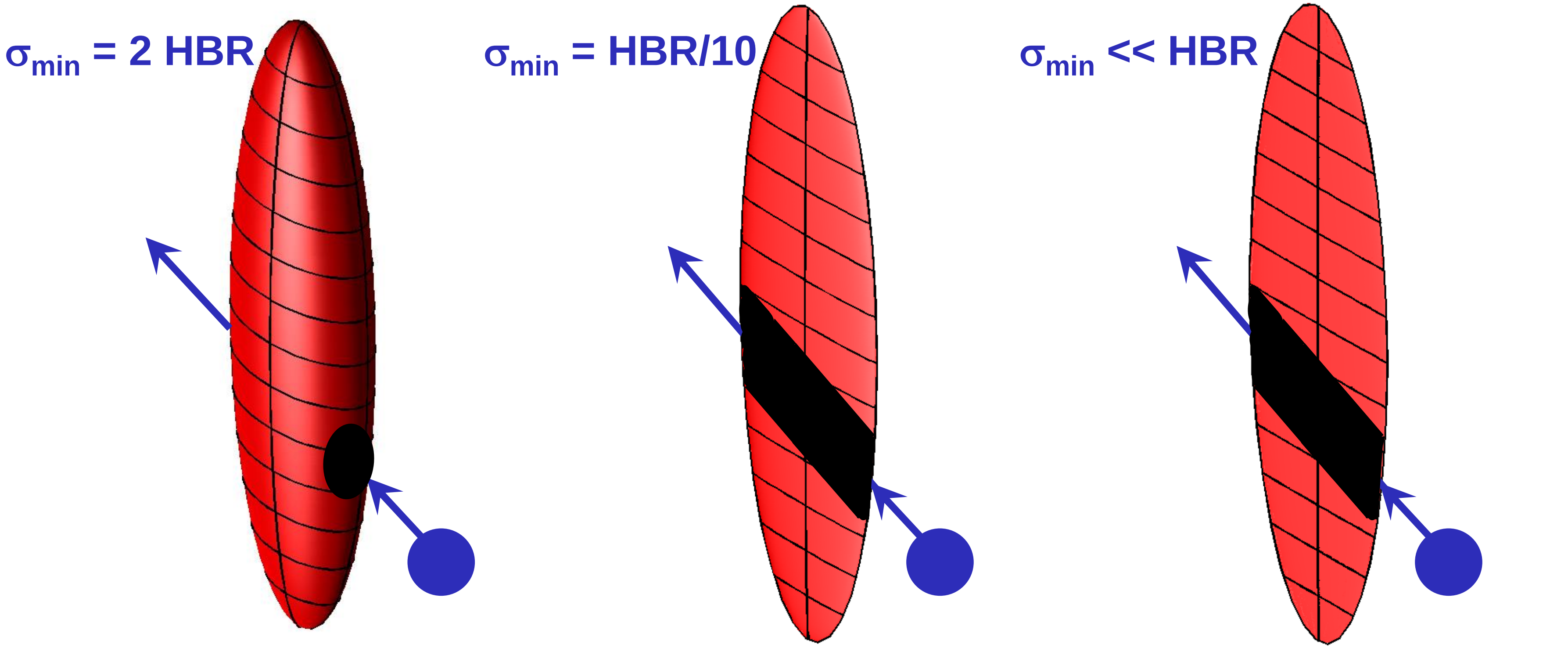
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



The two conjunctions on the right will produce similar Pc values
The one on the left will produce a smaller Pc value

Collision Probability Visualization: Pc as a Function of σ_{min}

The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



Pc values are insensitive to the σ_{min} value whenever $0 < \sigma_{min} \ll$ HBR
This can be used to set a sensible eigenvalue clipping level

Conclusions

- The frequency of NPD state covariance matrices decreased markedly in mid-2016
  - After increasing the covariance precision
- NPD state covariances occur much more frequently for objects in high-eccentricity orbits
  - Likely due to covariance interpolation inaccuracies
- Estimating Pc values doesn’t always require fully positive definite state covariances
  - Because the calculations use marginalized covariances
- Remediating ECI state covariances doesn’t change Pc values
  - Three remediation methods produce equivalent results
- Eigenvalue clipping can be used for remediation
  - It is the simplest to implement, and can be constrained when required using physics-based considerations